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# $r$-Special Subschemes and an Argument of Severi's*

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TO THE MEMORY OF NORMAN LEVINSON, WHO UNDERSTOOD LIFE AND APPRECIATED MATHEMATICS WITH CLASSICAL ROOTS

In the classical literature, there are numerous discussions of divisors with degree $n$ and projective dimension $\geqslant r$ on a smooth curve of genus $g$. Brill and Noether [3, pp. 290–293] were probably the first to assert in effect that these divisors form a set $H_n^r$ whose components each have dimension exactly $\tau + r$, with $\tau = (r + 1)(n - r) - rg$, if the curve has general moduli. Their argument, however, seems to show only that each component has dimension $\geqslant \tau + r$ no matter what the moduli are, provided one such divisor does exist [16, Remark 6, p. 168]. Such divisors are supposed to exist for $\tau \geqslant 0$, but the existence was proved only recently (in [10, 18] for $r = 1$ and the complex numbers, and in [13, 15, 16] for $r \geqslant 1$ and an arbitrary algebraically closed ground field). Moreover, it was recently proved [17] that the set $H_n^r$ does have at least one component with dimension $\tau + r$ for $r = 1, 2, 3$ and the complex numbers, if the curve has general moduli. Of course, there are special cases in which $H_n^r$ has dimension $> \tau + r$, for example, a hyperelliptic curve of genus $\geqslant 3$ for $n = 2$ and $r = 1$.

The goal (Theorem 5.3) of this article is a reduction of the above assertion of Brill and Noether to the following concrete geometric conjecture.

*Conjecture.* The $[n - r - 1]$-planes intersecting $g$ general secants of the smooth rational curve with degree $n$ in projective $n$-space are parameterized by a (closed) subset $T$ of the grassmannian with $\dim(T) \leqslant \tau$, with $\tau = (r + 1)(n - r) - rg$. Moreover, those $[n - r - 1]$-planes which in addition contain $d$ or more of the secants are parameterized by a (closed) subset with dimension $\leqslant \tau - (r + 2)d$.

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The reduction follows some of the broad lines of Severi's "proof" [23, Anhang G, Sect. 8, pp. 380–390] of Brill and Noether's assertion. Severi uses the first part of this conjecture, which he considers to be a fact. He says [23, very bottom of p. 381] that it holds because (1) the $[n - r - 1]$-planes intersecting $g$ general lines in $n$-space form a set whose components each have dimension $\tau$ (this statement is true and easy to verify) and (2) the $g$ general lines can be deformed continuously into the $g$ secants, and during such a deformation the set of corresponding $[n - r - 1]$-planes traverses the entire grassmannian, which is irreducible. Later, in a discussion of a related argument of Castelnuovo's [4], the same first part of the conjecture comes up again, and Severi says [23, next to the last paragraph on p. 393] that it holds because the $[n - r - 1]$-planes intersecting $g$ variable lines form an irreducible set. (This statement is easy to verify along the lines of (2) above, but how does it imply the first part of the conjecture?)

The conjecture is discussed by Dan Laksov in the Appendix. The second part of the conjecture is shown to follow from the first part with $n - 2d$ for $n$ and $g - d$ for $g$. Also, the conjecture is established for $r = 1$. Thus, for $r = 1$, Brill and Noether's assertion is completely proved below. This is the most important case of their assertion, and the proof gives the general flavor of our adaptation of Severi's argument. In the Appendix it is proved, moreover, that for $r = 1$ and characteristic zero, the components of $T$ each appear with multiplicity one, and that consequently, as Castelnuovo [4] asserted (cf. [16, p. 164]), on a curve of genus $g$ with general moduli the number of distinct linear series $g_n^1$ with $2(n - 1) = g$ is exactly $g!/(g - n + 1)!\,(g - n + 2)!$ These two assertions also can be established in other ways. For example, they are simple consequences of the following fact, which was pointed out by M. Artin in a conversation: The functor of $n$-fold coverings of $\mathbb{P}_{\mathbb{Z}}^1$ is unobstructed and has relative dimension $2n + 2g - 5$.

Severi's "proof" [23, pp. 381–383] of the key inequality, $\dim H_n^r \leqslant \tau + r$, runs roughly as follows. The curve with general moduli $X$ is degenerated into a rational curve $Y$ with $g$ nodes $x_1, \ldots, x_g$. By upper semicontinuity of dimension, it suffices to prove the corresponding inequality for $Y$. Let $Y'$ denote the normalization of $Y$, embedded with degree $n$ in $n$-space. A divisor with degree $n$ on $Y$ defines a hyperplane section of $Y'$. If the divisor contains the $i$th node $x_i$, then the hyperplane contains the two points $y_i$, $z_i$ lying over $x_i$. Hence an $r$-dimensional linear series of divisors with degree $n$ defines an $r$-parameter linear family of hyperplanes whose axis, an $[n - r - 1]$-plane, meets the $g$

secants $\overline{y_1 z_1}, \ldots, \overline{y_g z_g}$ of $Y'$. Finally, the first part of the conjecture yields the inequality.

There are two main difficulties. First, many different divisors on $Y$ give rise to the same divisor on $Y'$. This matter is considered in Sections 2 and 3 below, and it turns out that the set of linear series on $Y$ giving rise to the same family of hyperplanes has dimension $\leqslant d$ if the axis contains $\leqslant d$ of the $g$ secants. This bound is adequate if the second part of the conjecture is assumed (see the proof of (5.1a)).

The second difficulty is that specializing a divisor on $X$ need not yield a divisor on $Y$; the variety of divisors on $Y$ is not complete. Therefore, in Sections 3 and 4 there is developed a theory of "$r$-special" subschemes, ones with "projective dimension" $\geqslant r$. Section 4 is especially concerned with $r$-special subschemes that are not divisors. In particular, Theorem 4.6 implies that the set of $r$-special subschemes with degree $n$ on $Y$ that are not divisors at the $i$th node $x_i$ may be canonically identified with the set $H_{n-1}^r$ of $r$-special subschemes with degree $n-1$ on the curve $Y_i$ constructed from $Y$ by normalizing at $x_i$. By induction on $n$ or $g$, the set $H_{n-1}^r$ may be assumed to have dimension $\leqslant \tau + r - 1$. Consequently, the set $H_n{}^r$ of $r$-special subschemes with degree $n$ on $Y$ has dimension $\leqslant \tau + r$, and hence, so does the original set $H_n{}^r$ of $r$-special divisors with degree $n$ on $X$, if the full conjecture is assumed.

It remains to prove that each component of the set $H_n{}^r$ of $r$-special divisors with degree $n$ on $X$ has dimension $\geqslant \tau + r$. This is relatively easy to do (see the proof of Theorem 5.3). The principal step is carried out in Section 1 for an arbitrary Gorenstein curve using an abstract version of Brill and Noether's argument. Curiously, Severi [23, pp. 384–387] works equally hard on this estimate as on the other.

*Blanket notation.* Whenever $D$ is a closed subscheme of a scheme, its sheaf of ideals will be denoted $I(D)$.

Whenever $X$ is an $S$-scheme and $s$ is a point of $S$, the fiber of $X$ over $\mathrm{Spec}(k(s))$ will be denoted $X(s)$.

## 1. THE SUBSCHEME PARAMETERIZING THE $r$-SPECIAL SUBSCHEMES OF A FAMILY

DEFINITION 1.1. Let $k$ be a field and $X$ be a projective $k$-scheme. Let $C$ be a closed subscheme of $X$. Set

$$r(C) = \dim_k \mathrm{Hom}_{O_X}(I(C), O_X) - 1$$
$$(= \dim_k H^0(X, \underline{\mathrm{Hom}}_{O_X}(I(C), O_X)) - 1),$$

$$i(C) = \dim_k \operatorname{Ext}^1_{O_X}(I(C), O_X)$$
$$(= \dim_k H^1(X, \underline{\operatorname{Hom}}_{O_X}(I(C), O_X)) \text{ if } C \text{ is a divisor}).$$

(Suppose $C$ is a divisor. Then, of course, $r(C)$ is called the *projective dimension* of $C$, and $i(C)$ is called the index of specialty of $C$ if $X$ is a curve and the *superabundance* of $C$ if $X$ is a surface.) Let us call $C$ *r-special*, for an integer $r \geqslant 0$, if $r(C) \geqslant r$ holds.

LEMMA 1.2. *Let $p: X \to S$ be a projective morphism of locally noetherian schemes.*

(i) *Let $F$ be a coherent $O_X$-Module, flat over $S$. Then there exists a coherent $O_S$-Module $Q(F)$ and an element $q(F)$ in $H^0(X, F \otimes p^*Q(F))$ such that the Yoneda map,*

$$y(q(F)) : \operatorname{Hom}(Q(F), M) \to H^0(X, F \otimes p^*M)$$

*is an isomorphism for each quasi-coherent $O_S$-Module $M$. (In other words, the pair $(Q(F), q(F))$ represents the functor, $M \mapsto H^0(X, F \otimes p^*M)$, on the category of quasi-coherent $O_S$-Modules.) Moreover, the formation of the pair $(Q(F), q(F))$ commutes with base change.*

(ii) *Let $F$ and $G$ be coherent $O_X$-Modules, $F$ flat over $S$. Then there exists a coherent $O_S$-Module $H(G, F)$ and an element $h(G, F)$ in* $\operatorname{Hom}(G, F \otimes p^*H(G, F))$ *such that the Yoneda map,*

$$y(h(G, F)) : \operatorname{Hom}(H(G, F), M) \to \operatorname{Hom}(G, F \otimes p^*M),$$

*is an isomorphism for each quasi-coherent $O_S$-Module $M$. Moreover, the formation of the pair $(H(G, F), h(G, F))$ commutes with base change.*

(iii) *Let $F$ and $G$ be coherent $O_X$-Modules, $F$ be flat over $S$, and $G$ be locally free. Let $M$ be an invertible $O_S$-Module. Then there exist canonical isomorphisms,*

$$H(G, F) = Q(\underline{\operatorname{Hom}}(G, F)),$$
$$Q(F \otimes p^*M) = Q(F) \otimes M^{-1}.$$

*Proof.* The second formula of (iii) obviously holds. The remaining assertions are discussed in [2, (12), (13), (14)]. The existence of $(Q(F), q(F))$ and $(H(G, F), h(G, F))$ and the commutativity with base change are virtually proved in [9, EGA $\mathrm{III}_2$, 7.7.6, 7.7.8, and 7.7.9].

(1.3) Let $p: X \to S$ be a flat, projective morphism of locally noetherian schemes. Let $D$ be a closed subscheme of $X$, flat over $S$. Fix an integer $r \geqslant 0$.

DEFINITION. The closed subscheme of $S$, defined by the $(r+1)$st Fitting ideal [21, Sect. 5, p. 114] of $H(I(D), O_X)$ (see Lemma 1.2(ii)), will be denoted by

$$[S; D, X]^r \qquad \text{or} \qquad [S]^r.$$

PROPOSITION. (i) *The underlying set of* $[S]^r$ *consists of those* $s \in S$ *such that* $D(s)$ *is an r-special subscheme of* $X(s)$.

(ii) *The formation of* $[S]^r$ *commutes with base change.*

*Proof.* The formation of $[S]^r$ commutes with base change because the formations of a Fitting ideal [21, 5.1(a), p. 114], of an $H(G, F)$, and of the ideal of a flat subscheme all do. It is now a straightforward matter to verify that $[S]^r$ has the described underlying set by reducing first to the case that $S$ is Spec of a field.

PROPOSITION 1.4. *Let* $p: X \to S$ *be a flat, projective morphism of locally noetherian schemes. Let* $T$ *be a scheme proper over* $S$ *and* $D$ *be a closed subscheme of* $X \times_S T$ *flat over* $T$. *Fix an integer* $r \geqslant 0$. *Then the function,*

$$s \mapsto \dim[T(s); D(s), X \times_s T(s)]^r,$$

*is upper semicontinuous on* $S$.

*Proof.* The assertion results from the theorem on the upper semicontinuity of the dimensions of the fibers of a proper morphism because $[T; D, X \times T]^r$ is, by (1.3), a closed subscheme of $T$, whose formation commutes with base change.

THEOREM 1.5. *Let* $p: X \to S$ *be a flat, projective morphism of locally noetherian schemes. Let* $D$ *be a closed subscheme of* $X$, *flat over* $S$. *Fix an integer* $r \geqslant 0$.

(i) *Assume* $D$ *is a divisor. Then, for each* $s \in [S]^r$, *there is an upper bound on the codimension of* $[S]^r$ *at* $s$, *namely,*

$$\operatorname{codim}_s([S]^r, S) \leqslant r(r - r(D(s)) + i(D(s))).$$

(ii) *Assume the fibers* $X(s)$ *are Gorenstein curves of arithmetic genus* $g$ *such that* $H^0(O_{X(s)}) = k(s)$ *and assume the fibers* $D(s)$ *have degree* $n$ *(that is,* $n = \dim H^0(O_{D(s)})$ *holds). Then there is an upper bound on the codimension of* $[S]^r$ *valid at every* $s \in [S]^r$*, namely,*

$$\operatorname{codim}_s([S]^r, S) \leqslant r(r - n + g).$$

(Note that, by Riemann's theorem, parts (i) and (ii) give the same bound when they both apply.)

*Proof.* (i) The inclusion map, $u: I(D) \to O_X$, corresponds to a map,

$$v: H(I(D), O_X) \to O_S.$$

For each $s \in S$, the fiber $u(s)$ is nonzero and it corresponds by the Yoneda map (see Lemma 1.2(ii)) to the fiber $v(s)$; hence, $v(s)$ is nonzero. So $v(s)$ is surjective because its target is $k(s)$. Therefore, by Nakayama's lemma, $v$ is surjective.

Since $D$ is a divisor, $I(D)$ is invertible. Hence, by Lemma 1.2(iii), there is a canonical isomorphism,

$$H(I(D), O_X) = Q(L), \qquad \text{with} \qquad L = \underline{\mathrm{Hom}}(I(D), O_X).$$

Fix $s \in S$, and let $K^{\cdot}$ be a finite cohomology complex for $L$ at $s$. It is a complex of free Modules on an affine neighborhood $U$ of $s$, accompanied by an isomorphism,

$$H^i(K^{\cdot} \otimes M) = H^i(p^{-1}U, L \otimes p^*M),$$

which is functorial in the quasi-coherent $O_U$-Module $M$. (Its existence is established in [20, 6.10.5, p. 41] as well as in [9].) Then there is an exact sequence,

$$K^{1*} \to K^{0*} \to Q(L) \to 0,$$

where $K^{1*}$ and $K^{0*}$ denote the dual Modules. Assume the complex $K^{\cdot}$ is minimal at $s$, so that the following relation holds:

$$\operatorname{rank}(K^i) = \dim_{k(s)}(H^i(L(s))).$$

Let $R$ be the Kernel of $v$. Then splitting off direct summands from

$Q(L)$ and $K^{0*}$, shrinking $U$ about $s$ if necessary, yields an exact sequence on $U$,

$$K^{1*} \xrightarrow{w} W \longrightarrow R \longrightarrow 0,$$

and $W$ is free with rank $r(D(s))$ and $K^{1*}$ is free with rank $i(D(s))$. Clearly, $[S]^r \cap U$ is the scheme of zeros of $\wedge^a w$ with $a = r(D(s)) - r$, or empty if $a \leqslant 0$. It is well-known (see [14, Corollary 11]) that a zero set of this sort has codimension $\leqslant r(i(D(s)) - a)$.

(ii) Since $X/S$ has Gorenstein fibers, it has an invertible dualizing sheaf $\omega$ [11, Exercise 9.7, p. 298]. Then the natural sequence,

$$0 \to \omega \otimes I(D) \to \omega \to \omega \otimes O_D \to 0,$$

is exact and has all terms flat over $S$. So there is a derived exact sequence,

$$p_*\omega \xrightarrow{w} p_*\omega \otimes O_D \longrightarrow R^1p_*\omega \otimes I(D) \longrightarrow R^1p_*\omega \longrightarrow 0.$$

Its formation commutes with base change; moreover, the first and second terms are locally free with ranks $g$ and $n$, and the fourth term is equal to $O_S$.

Indeed, the second term is locally free with rank $n$ and its formation commutes with base change because $\omega \otimes O_D$ is flat over $S$ and its support $D$ is finite over $S$ with degree $n$. The formation of the third and fourth terms commutes with base change because the fibers of $p$ are curves (see [19, Corollary 1, p. 51]). The trace map carries the fourth term isomorphically onto $O_S$ because its formation commutes with base change and by virtue of the hypotheses it is an isomorphism on fibers. Finally, the first term is therefore locally free and its formation therefore commutes with base change by [19, (ii, iii), p. 51]; it has rank $g$ because it does on the fibers.

For each $s \in S$, duality [11, Remark 2, p. 214; 1, (1.3), p. 5] yields

$$H^0(X(s), \omega \otimes I(s)) = \mathrm{Hom}_{O_{X(s)}}(I(s), O_{X(s)})^*$$

because $\omega$ is invertible and its formation commutes with base change. It follows now that the underlying set of $[S]^r$ consists of the zeros of $\wedge^{n-r} w$. (More work yields this scheme-theoretically.) Therefore, $[S]^r$ has codimension $\leqslant r(g - n + r)$ in $S$ at $s$.

*Remark* 1.6. Theorem 1.5(ii) will be used only for a constant family $X/S$; that is, $X$ will arise by base change from a curve $X_0$ over a field.

In this case, $\omega$ can be taken directly as the pull-back of the dualizing sheaf on $X_0$, and the duality theory that is needed comes directly from that for $X_0$ (as presented in [1], for example).

## 2. The Map between the Schemes of Effective Divisors

Proposition 2.1. *Let $f\colon X' \to X$ be a morphism of schemes, flat and projective over a locally noetherian base scheme $S$. Assume the associated points of the fibers of $X'/S$ are carried to associated points of the fibers of $X/S$.*

(i) *The morphism $f$ induces a well-defined $S$-morphism on the schemes of effective divisors,*

$$f^*\colon \mathrm{Div}_{(X/S)} \to \mathrm{Div}_{(X'/S)}\,.$$

(ii) *If there is an open subset $U$ of $X$ such that the restriction, $f^{-1}U \to U$, is an isomorphism, then the open subscheme $\mathrm{Div}_{(U/S)}$ of $\mathrm{Div}_{(X/S)}$ is the full inverse image of the open subscheme $\mathrm{Div}_{(f^{-1}U/S)}$ of $\mathrm{Div}_{(X'/S)}$ and the restriction of $f^*$ is an isomorphism.*

*Proof.* (i) Let $T$ be a locally noetherian $S$-scheme and $D$ be a relative effective divisor on $X \times T/T$. It is necessary to prove that the subscheme $f_T{}^*D$ of $X \times T/T$ is a relative (effective) divisor. Now, a fiber $(f_T{}^*D)(t)$ is obviously equal to the inverse image $f_T(t)^*D(t)$, so it is a divisor by [9, EGA $\mathrm{IV}_4$, 21.4.4, ii] in view of the hypothesis about associated points. Furthermore, $f_T{}^*D$ is obviously defined locally by one equation, and $X \times T/T$ is flat. Hence, $f_T{}^*D$ is a relative divisor by [1, VIII, 4.1, p. 142].

(ii) The assertion is obvious.

Lemma 2.2. *Let $f\colon X' \to X$ be a morphism of schemes that are flat and projective over a locally noetherian base scheme $S$. Assume that the associated points of the fibers of $X'/S$ are carried to associated points of the fibers of $X/S$. Assume $X'/S$ is flat. Assume the formation of $f_*O_{X'}$ commutes with base change. Assume the comorphism, $O_X \to f_*O_{X'}$, is universally injective (it will be thought of as an inclusion). Fix a relative effective divisor $D$ on $X/S$.*

(i) *There exists a natural map $\Phi$ from the set of relatively effective* *divisors $\Delta$ on $X/S$ satisfying $f^*\Delta = f^*D$ into the set $\Gamma_D(X, f_*O^*_{X'}/O_X{}^*)$ of global sections with supports in $D$ (where, as usual, * indicates the subsheaf of invertible elements).*

*The map $\Phi$ is compatible with base change, and it is universally injective.*

(ii) *Assume the following relation holds:*

$$I(D) \cdot \Gamma_D(X, f_*O_{X'}/O_X) = 0.$$

*Then the map $\Phi$ in* (i) *is universally bijective.*

*Proof.* Fix $x \in X$. Fix a local defining element $a \in O_{X,x}$ for $D$ at $x$. Let $\Delta$ be a relative effective divisor on $X/S$ satisfying $f^*\Delta = f^*D$ and fix a local defining element $\alpha \in O_{X,x}$. The subschemes $f^*\Delta$ and $f^*D$ are divisors by Proposition 2.1(i); hence, $\alpha$ and $a$ are nonzero divisors in a neighborhood of $f^{-1}(x)$, and so they belong to $(f_*\mathscr{M}^*_{X'})_x$, where $\mathscr{M}_{X'}$, denotes the sheaf of meromorphic functions on $X'$. The exact sequence (induced by the inclusion of $O^*_{X'}$ into $\mathscr{M}^*_{X'}$),

$$0 \to f_*O^*_{X'} \to f_*\mathscr{M}^*_{X'} \to f_*(\mathscr{M}^*_{X'}/O^*_{X'}),$$

now shows that there exists a unique element $u \in (f_*O^*_{X'})_x$ such that

$$\alpha = ua$$

holds. Since $\alpha$ and $a$ are determined up to elements of $O^*_{X,x}$, the class in $(f_*O^*_{X'}/O_X{}^*)_x$ of $u$ is well-determined by $\Delta$ (with $D$ fixed). It is evident that, as $x$ varies, these classes form an element $\Phi(\Delta)$ of $\Gamma(X, f_*O^*_{X'}/O_X{}^*)$ and that the formation of $\Phi(\Delta)$ from $\Delta$ commutes with base change.

Suppose $x \notin D$. Then $a$ is a unit. Hence $u$ lies in the intersection,

$$(f_*O^*_{X'})_x \cap O_{X,x}.$$

This intersection is equal to $O^*_{X,x}$ because $f_*O_{X'}$ is a finite $O_X$-Module as $f$ is projective. Thus $u$ lies on $O^*_{X,x}$. (Specifically, an integral equation for $u^{-1}$ yields an expression for $u^{-1}$ as a polynomial in $u$.) Therefore $\Phi(\Delta)$, which is defined by $u$ at $x$, has support in $D$.

Obviously, $u$ determines $\alpha$ if $a$ is fixed, and so the class in $(f_*O^*_{X'}/O_X{}^*)_x$ of $u$ determines $I(\Delta)_x$ as $D$ is fixed. Therefore, the map $\Phi$ sending $\Delta$ to $\Phi(\Delta)$ is injective, and so universally injective.

Assume the relation in (ii) holds. Take an element,

$$s \in \Gamma_D(X, f_*O^*_{X'}/O_X{}^*).$$

At $x$, represent $s$ by an element $u$ of $(f_*O^*_{X'})_x$. Set $\alpha = ua$. The relation in (ii) implies that $\alpha$ lies in $O_{X,x}$. Obviously $\alpha$ is a nonzero divisor there. Obviously $\alpha$ is determined up to an element of $O^*_{X,x}$ by $s$ and $D$. So, as $x$ varies, the corresponding $\alpha$'s define an effective divisor $\Delta_1$ on $X$.

Obviously the formation of $\Delta_1$ from $s$ and $D$ commutes with base change. In particular, $\Delta_1$ remains a divisor after base change. Hence $\Delta_1$ is a relative effective divisor. Obviously $\Phi(\Delta_1)$ is equal to $s$. Thus $\Phi$ is surjective. Obviously the relation in (ii) is stable under base change because the formation of $f_*O_{X'}$ commutes with base change. Therefore $\Phi$ is universally surjective, and so universally bijective.

*Remark* 2.3. The map $\Phi$ of Lemma 2.2(i) is easily proved compatible with the passage to invertible sheaves in the sense expressed by the formula,

$$O_X(\Delta) = \delta(\Phi(\Delta)) \otimes O_X(D),$$

where $\delta\colon \Gamma(X, f_*O^*_{X'}/O_X{}^*) \to H^1(X, O_X{}^*)$ is the boundary map (or its negative).

THEOREM 2.4. *Let $X$ be a reduced, projective curve over a field $k$. Let $f\colon X' \to X$ denote the normalization map.*

(i) *The map $f$ induces a well-defined, birational map between the schemes of effective divisors,*

$$f^*\colon \mathrm{Div}_{(X/k)} \to \mathrm{Div}_{(X'/k)}\,.$$

(ii) *Let $D$ be an effective divisor on $X$ whose support contains exactly $d$ ($d \geqslant 0$) nodes (that is, rational ordinary double points) and no other singularities. Then there is a canonical isomorphism of $k$-schemes,*

$$(f^*)^{-1} f^*(D) = G_m^{\times d}.$$

*Proof.* (i) The hypotheses of Proposition 2.1 are obviously satisfied, with $U$ taken to be the set of normal points of $X$. So $f^*$ is well-defined, and it is birational because the open subschemes $\mathrm{Div}_{(U/k)}$ of $\mathrm{Div}_{(X/k)}$ and $\mathrm{Div}_{(f^{-1}U/k)}$ of $\mathrm{Div}_{(X'/k)}$ are, obviously, dense.

(ii) The comorphism, $O_X \to f_*O_{X'}$, is injective by construction. So it is universally injective and the formation of $f_*O_{X'}$ commutes with base change because every base change is flat since $k$ is a field. The relation,

$$I(D) \cdot \Gamma_D(X, f_*O_{X'}/O_X) = 0,$$

holds because the annihilator of the $O_X$-Module $f_*O_{X'}/O_X$ is equal to the maximal ideal at a node of $X$ and it is equal to $O_X$ at a simple point of $X$. The remaining hypotheses of Lemma 2.2 are obviously satisfied.

Rephrased, Lemma 2.2 asserts that, for each locally noetherian $k$-scheme $T$, the $T$-points of $(f^*)^{-1}f^*(D)$ may be $T$-functiorially identified with the set,

$$\Gamma_{D_T}(X_T, (f_T)_* O^*_{X'_T}/O^*_{X_T}).$$

It remains to identify this set functorially with the set $\Gamma(T, O_T{}^*)^{\times r}$ of $T$-points of $G_m^{\times r}$.

Let $x$ be a node of $X$, and $y$, $z$ the two points of $X'$ lying over it. The natural maps, $O_{X'} \to k(y)$ and $O_{X'} \to k(z)$, clearly induce an isomorphism of sheaves of algebras

$$(\sigma, \tau): f_*O_{X'}/I(x) \xrightarrow{\sim} k(x) \oplus k(x),$$

which carries $O_X/I(x)$ onto the diagonal. Hence there is an isomorphism,

$$(\sigma/\tau): (f_*O^*_{X'}/O_X{}^*) \xrightarrow{\sim} k(x)^*.$$

induced by dividing the values of $\sigma$ by those of $\tau$. Clearly, this construction is compatible with base change, yielding, for each $k$-scheme $T$, an isomorphism,

$$(\sigma_T/\tau_T): (f_T)_* O^*_{X_T'}/O^*_{X_T} \xrightarrow{\sim} O^*_{x\times T},$$

which is compatible with further base change. Putting together the $r$ such isomorphisms, one for each node in the support of $D$, yields the functorial identification sought on global sections.

## 3. The Linear Maps Covering the Map Between the Picard Schemes

Lemma 3.1. *Let $X$ be a flat, projective scheme with geometrically integral fibers and a fixed section over a locally noetherian base scheme $S$.*

(i) *The Picard scheme $P = \mathrm{Pic}_{(X/S)}$ exists and is a quasi-projective $S$-scheme, locally of finite type.*

(ii) *The product $X \times_S P$ carries a Poincare (or universal invertible)* *sheaf $L$, which is unique when normalized by being made trivial along the section of $X \times_S P/P$ induced by the section of $X/S$.*

(iii) *Referring to Lemma* 1.2(i), *set $E = Q(L)$. Then there exists a canonical isomorphism of $P$-schemes,*

$$\mathrm{Div}_{(X/S)} = \mathbb{P}(E).$$

(iv) *The universal divisor on $X \times_S \mathrm{Div}_{(X/S)}$ corresponds to the section of the tensor product of the pull-backs of $L$ and $O_{\mathbb{P}(E)}(1)$ determined (see Lemma* 1.2(i)) *by the fundamental surjection, $E_{\mathbb{P}(D)} \twoheadrightarrow O_{\mathbb{P}(E)}(1)$.*

*Proof.* Proofs are outlined in [8]. Detailed proofs for generalizations of (iii) and (iv) are given in [2, (15)].

PROPOSITION 3.2. *Keep the setup of Lemma* 3.1. *Fix an integer $r \geqslant 0$. Let $P_r$ denote the subscheme of $P = \mathrm{Pic}_{(X/S)}$ defined by the $(r+1)$st Fitting ideal of $E = Q(L)$.*

(i) *There exists a canonical isomorphism of $P$-schemes,*

$$[\mathrm{Div}_{(X/S)}]^r = \mathbb{P}(E \mid P_r).$$

(ii) *The structure map of the $P$-scheme $\mathrm{Grass}_{r+1}(E)$ factors through a surjection,*

$$\mathrm{Grass}_{r+1}(E) \to P_r\,,$$

*which is an isomorphism exactly over $(P_r - P_{r+1})$.*

(iii) *Let $F$ denote the universal quotient of $E$ on $\mathrm{Grass}_{r+1}(E)$. The natural map from $\mathbb{P}(F)$ into $\mathbb{P}(E) = \mathrm{Div}_{(X/S)}$ factors through a surjection,*

$$\mathbb{P}(F) \to [\mathrm{Div}_{(X/S)}]^r,$$

*which is an isomorphism exactly off $[\mathrm{Div}_{(X/S)}]^{r+1}$.*

*Proof.* (i) It is easy to see that $[\mathrm{Div}_{(X/S)}]^r$ is defined by the $(r+1)$st Fitting ideal of $Q(L_{X \times \mathbb{R}(E)}) \otimes O_{\mathbb{P}(E)}(-1)$ by using the definition in (1.3), the description in Lemma 3.1(iv) and both formulas in Lemma 1.2(iii). On the other hand, $\mathbb{P}(E \mid P_r)$ is the fiber of $\mathbb{P}(E)$ over $P_r$; so it is defined by the $(r+1)$st Fitting ideal of $Q(L_{X \times \mathbb{P}(E)})$ because the formations of a Fitting ideal and of a $Q(F)$ commute with base change. Finally, the two $(r+1)$st Fitting ideals are obviously equal.

(ii) The $(r+1)$st Fitting ideal of the pull-back of $E$ to $\mathrm{Grass}_{r+1}(E)$

is obviously zero. Hence the structure map factors through a map, $\mathrm{Grass}_{r+1}(E) \to P_r$. This map is surjective because its fiber over a point $\pi \in P_r$ is equal to $\mathrm{Grass}_{r+1}(E(\pi))$ and $\dim(E(\pi))$ is at least $r + 1$. For a similar reason, the map cannot be bijective at any point of $P_{r+1}$. Finally, the map is an isomorphism over $(P_r - P_{r+1})$ because the restriction of $E$ is locally free with rank $r + 1$.

(iii) The assertion follows easily from (i) and (ii) with the aid of some obvious general facts.

PROPOSITION 3.3. *Let $X$ and $X'$ be flat, projective schemes with geometrically integral fibers and fixed sections over a locally noetherian base scheme $S$. Let $f: X' \to X$ be an $S$-morphism compatible with the sections. Assume the formation of $f_*O_{X'}$ commutes with base change and the comorphism, $O_X \to f_*O_{X'}$, is universally injective. Let $P$, $L$, $E$ and $P'$, $L'$, $E'$ denote the Picard schemes, the normalized Poincare sheaves, the associated sheaves (see Lemma* 3.1(iii)) *for $X$ and $X'$.*

(i) *There exists a natural surjection of $O_P$-Modules,*

$$u: E'_P \to E.$$

(ii) *The "linear" map, $\mathbb{P}(E) \to \mathbb{P}(E')$, defined by $u$ is equal to the map $f^*$ between the schemes of divisors.*

(iii) *Fix an integer $r \geq 0$. Let $F$ and $F'$ denote the universal quotient bundles on* $\mathrm{Grass}_{r+1}(E)$ *and* $\mathrm{Grass}_{r+1}(E')$. *The surjection $u$ yields the following diagram, in which the horizontal maps are closed embeddings:*

$$\begin{array}{ccc}
\mathbb{P}(F') & \longrightarrow & \mathbb{P}(E') \times_P \mathrm{Grass}_{r+1}(E') \\
\uparrow & & \uparrow \\
\mathbb{P}(F) \to \mathbb{P}(E) \times_P \mathrm{Grass}_{r+1}(E) & \to & \mathbb{P}(E) \times_{P'} \mathrm{Grass}_{r+1}(E').
\end{array}$$

*Proof.* (i) The pull-backs of $L$ and $L'$ to $X' \times_S P$ are equal. So the natural map, $L \to f_{P*}f_P{}^*L$, may be rewritten as

$$L \to f_{P*}(L'_{X' \times P}).$$

There is induced, for each quasi-coherent $O_P$-Module $M$, a map,

$$\Gamma(X \times P, L \otimes M_{X \times P}) \to \Gamma(X' \times P, L'_{X' \times P} \otimes M_{X' \times P}),$$

which is functorial in M. Hence, by Yoneda's lemma, this map of functors is represented by a map $u$ into $Q(L) = E$ from $Q(L'_{X' \times P})$. The latter module is equal to $E'_P$ by compatibility with base change.

It is easy to see that the Cokernel of $u$ will be zero at a point $\pi \in P$ if the induced map,

$$\mathrm{Hom}(Q(L) \otimes k(\pi), k(\pi)) \to \mathrm{Hom}(Q(L'_{X' \times P}) \otimes k(\pi), k(\pi)),$$

is injective. This map is clearly equal to the map on global sections of the natural map,

$$L(\pi) \to f_{\pi *} f_\pi^{\ *} L(\pi).$$

Hence all these maps are injective because the comorphism of $f_\pi$ is injective, in view of the hypotheses, and $L$ is locally free.

(ii) The assertion is obvious from the description of the universal divisor in Lemma 3.1(iv) and the construction of $u$ in (i) above.

(iii) The assertion is elementary. The second map on the bottom arises from the embedding of $\mathrm{Grass}_{r+1}(E)$ into $\mathrm{Grass}_{r+1}(E'_P)$, and the latter scheme is equal to $\mathrm{Grass}_{r+1}(E) \times_{P'} P$.

(3.4) Let $X$ be a rational (integral) projective curve with $g$ nodes over a field $k$. Let $f\colon X' \to X$ denote the normalization map. Fix compatible (rational) base points on $X'$ and $X$. Let $P, L, E$ and $P', L', E'$ denote the Picard schemes, the normalized Poincaré sheaves, the associated sheaves (see Lemma 3.1(iii)) for $X$ and $X'$. The Picard schemes decompose into disjoint unions of connected components $P_n$ and $P'_n$, which parameterize the invertible sheaves of degrees $n$.

The normalization $X'$ is isomorphic to $\mathbb{P}^1$. Hence, $P'_n$ is a single point, $L' \mid X' \times P'_n$ is isomorphic to $O_{\mathbb{P}^1}(n)$, and $(E' \mid P'_n)^*$ is isomorphic to $\Gamma(O_{\mathbb{P}^1}(n))$. Fix $n \geqslant 1$. Then there is an isomorphism,

$$\mathbb{P}((E' \mid P'_n)^*) = \mathbb{P}^n.$$

The canonical surjection, $E'^*_{X' \times P'} \to L'$, then defines a closed embedding,

$$X' \to \mathbb{P}^n,$$

and it is not hard to see using the description (Lemma 3.1(iv)) that the pullback of the universal hyperplane on $\mathbb{P}^n \times \mathbb{P}(E' \mid P'_n)$ is equal to the universal divisor with degree $n$ on $X' \times \mathbb{P}(E' \mid P'_n)$. Hence the map from $\mathbb{P}(E \mid P_n)$ into $\mathbb{P}(E' \mid P'_n)$ defined by the natural surjection,

$u: E_{p'} \to E$, (see Proposition 3.3(i, ii)) may be thought of as carrying divisors on $X$ to hyperplanes in $\mathbb{P}^n$.

Fix $r \geqslant 0$. There is a canonical isomorphism,

$$\mathrm{Grass}_{r+1}(E' \mid P'_n) \simeq \mathrm{Grass}_{n-r-1}(\mathbb{P}^n).$$

It carries an $(n-r)$-dimensional subspace $V$ of $E' \mid P'_n$ to the $(n-r-1)$-plane $\mathbb{P}(V^*)$ in $\mathbb{P}^n$. Preceding it with the map defined by the natural surjection, $u: E_{p'} \to E$, (see Proposition 3.3(i)) yields this very important map:

$$\gamma: \mathrm{Grass}_{r+1}(E \mid P_n) \to \mathrm{Grass}_{n-r-1}(\mathbb{P}^n).$$

Let $x_1, \ldots, x_g$ be the nodes of $X$, and $y_i$, $z_i$ the two points of $X'$ lying over $x_i$. Let $M_i$ denote the line in $\mathbb{P}^n$ determined by $y_i$ and $z_i$.

THEOREM. (i) *The image of* $\mathrm{Grass}_{r+1}(E \mid P_n)$ *under* $\gamma$ *lies in the subscheme of* $\mathrm{Grass}_{n-r-1}(\mathbb{P}^n)$ *parameterizing the* $[n-r-1]$*-planes meeting the* $g$ *secant lines* $M_1, \ldots, M_g$.

(ii) *Let* $N$ *be an* $[n-r-1]$*-plane meeting* $M_1, \ldots, M_g$ *but containing at most* $d (d \geqslant 0)$ *of them. Then there is an upper bound,*

$$\dim(\gamma^{-1}(N)) \leqslant d.$$

*Proof.* For clarity, the proofs implicitly assume there are enough rational points, planes, etc. However, it is evident that the ground field may be assumed algebraically closed without loss of generality.

(i) Consider an $[n-r-1]$-plane $N$ in the image. It is easy to see using the description above of the map from $\mathbb{P}(E \mid P_n)$ into $\mathbb{P}(E' \mid P'_n)$ that each hyperplane containing $N$ comes from a divisor on $X$. Hence, if the hyperplane contains $y_i$, it also contains $z_i$ because the divisor must contain $x_i$. Therefore the linear span $H$ of $N$ and $y_i$ contains $z_i$. So $H$ contains $M_i$. Since the codimension of $N$ in $H$ is at most one, $N$ meets $M_i$.

(ii) There exists a hyperplane $H$ containing $N$ but containing at most $d$ of the $M_i$. Obviously any divisor on $X$ mapping to $H$ can contain at most $d$ of the $x_i$. So by Theorem 2.4(ii) the divisors on $X$ mapping to $H$ form a family with dimension at most $d$. Hence, in view of Proposition 3.3(iii), the inverse image in $\mathbb{P}(F)$ of the point in $\mathbb{P}(F')$ defined by $(H, N)$ has dimension at most $d$. Therefore $\mathbb{P}(F \mid \gamma^{-1}(N))$

has dimension at most $d + r$ because $\mathbb{P}(F' \mid N)$ has dimension at most $r$. Consequently, $\gamma^{-1}(N)$ has dimension at most $d$.

*Remark* 3.5. In (Theorem 3.4(ii)) the fiber $\gamma^{-1}(N)$ is isomorphic to $G^m_{\times d}$ if $N$ meets $M_1, \dots, M_g$ and contains exactly $d$ of them, say, $N_1, \dots, Nr$. The isomorphism is obtained by fixing a point in the fiber. The point determines an $r$-dimensional linear system on $X$, which is carried divisor by divisor linearly onto another one by the action of an element of

$$\Gamma_{\{x_1,\dots,x_d\}}(f_* O^*_{X'}/O_X{}^*) = G_m^{\times d}(k).$$

It is not hard to develop formal proofs of these statements (see Remark 2.3 and the proof of Theorem 2.4(ii)).

## 4. $r$-Special Subschemes that Are Not Divisors

LEMMA 4.1. *Let $X$ be a projective scheme over a locally noetherian base scheme $S$. Let $F$ and $G$ be coherent $O_X$-Modules, $F$ flat over $S$. Let $u: G \to F$ be an $O_X$-homomorphism. Then there exists a* (*unique*) *closed subscheme $Z_S(u)$ of $S$ of zeros of $u$; that is, a morphism $T \to S$ factors through $Z_S(u)$ if and only if the relation $u_T = 0$ holds.*

*Proof.* Apply Lemma 1.2(iii). The map $u: G \to F$ corresponds to a map $v: H(G, F) \to O_S$. The image of $v$ is an ideal and defines a closed subscheme of $S$. This subscheme has the desired property because the correspondence between $u$ and $v$ commutes with any base change $T \to S$ and $u_T$ is zero if and only if $v_T$ is.

PROPOSITION 4.2. *Let $X$ be a projective scheme over a locally noetherian base scheme $S$. Let $A$ be a coherent $O_X$-Algebra, and set $C = \mathrm{Hom}_{O_X}(A, O_X)$. Assume* (a) *the formation of $C$ commutes with base change,* (b) *the dual $c: C \to O_X$ of the structure map $O_X \to A$ is injective, and* (c) *the sheaf* Coker($c$) *is flat over $S$. Then there exists a natural closed subscheme $Y$ of* $\mathrm{Hilb}_{(X/S)}$ *parameterizing the flat, closed subschemes of $X$ whose ideals are $A$-Modules.*

*Proof.* Let $W$ denote the universal subscheme on $X \times \mathrm{Hilb}_{(X/S)}$. Consider the natural map,

$$w: I(W) \to \mathrm{Coker}(c)_{\mathrm{Hilb}},$$

arising from the inclusion of $I(W)$ into $O_{X \times \mathrm{Hilb}}$. Since $\mathrm{Coker}(c)$ is flat, $w$ has a closed subscheme $Z$ of $\mathrm{Hilb}_{(X/S)}$ of zeros by Lemma 4.1.

The hypotheses imply that there is an exact sequence on $X \times Z$,

$$0 \to \underline{\mathrm{Hom}}(A_Z, O_{X \times Z}) \to O_{X \times Z} \to \mathrm{Coker}(c)_Z \to 0.$$

Hence, since $w_Z$ is zero, there is a natural (injective) map,

$$I(W)_Z \to \underline{\mathrm{Hom}}_{O_{X \times Z}}(A_Z, O_{X \times Z}).$$

Following it with the map induced by the natural map from $O_{X \times_S Z}$ onto $O_{W \times_{\mathrm{Hilb}} Z}$ yields a map,

$$I(W)_Z \to \underline{\mathrm{Hom}}_{O_{X \times Z}}(A_Z, O_{W \times_{\mathrm{Hilb}} Z}).$$

This map corresponds to a map,

$$I(W)_Z \otimes A_Z \to O_{W \times_{\mathrm{Hilb}} Z}.$$

Since $W \times_{\mathrm{Hilb}} Z$ is flat over $Z$, the last map has a closed subscheme $Y$ of $Z$ of zeros by Lemma 4.1.

Let $V$ be a flat, closed subscheme of $X \times T$, where $T$ is a locally noetherian $S$-scheme. Clearly $I(V)$ is an $A_T$-Module if and only if (1) $I(V)$ is contained in the ideal $\underline{\mathrm{Hom}}(A_T, O_{X \times T})$ of $O_{X \times T}$ and (2) the corresponding map,

$$I(V) \otimes A_T \to O_V$$

is zero. Clearly condition (1) holds if and only if the map $T \to \mathrm{Hilb}_{(X/S)}$ inducing $V$ from $W$ factors through $Z$. Assume (1) holds. Then clearly (2) holds if and only if the map $T \to Z$ factors through $Y$. Thus $Y$ parameterizes the subschemes of $X$ whose ideals are $A$-Modules.

LEMMA 4.3. *Let $X$ be a scheme over a base scheme $S$ and $A$ be an $O_X$-Algebra. Set $C = \underline{\mathrm{Hom}}_{O_X}(A, O_X)$. Assume that $C$ is an invertible $A$-Module and that the dual $C \to O_X$ of the structure map $O_X \to A$ is injective (it will be thought of as an inclusion). Then*:

(i) *The assignment $J \mapsto C \cdot J$ gives a well-defined bijection from the set of ideals $J$ of $A$ onto the set of ideals $I$ of $O_X$ that are $A$-Modules.*

(ii) *Assume the formation of $C$ commutes with base change and the map $C \to O_X$ remains injective after base change. Then the assignment*

$J \mapsto C \cdot J$ *of* (i) *commutes with base change in the sense expressed by the formula,*

$$C_T \cdot (JO_{X\times T}) = (C \cdot J)\, O_{X\times T} \qquad \text{for} \quad T/S.$$

(iii) *Let $J$ be an ideal of $A$. Then there is a canonical exact sequence,*

$$0 \to C \otimes_A (A/J) \to O_X/C \cdot J \to O_X/C \to 0.$$

(iv) *Assume $O_X/C$ is flat over $S$. Let $J$ be an ideal of $A$. Then $A/J$ is flat over $S$ if and only if $O_X/C \cdot J$ is.*

(v) *Let $J$ be an ideal of $A$. Then there is a canonical isomorphism.*

$$\underline{\mathrm{Hom}}_A(J, A) = \underline{\mathrm{Hom}}_{O_X}(C \cdot J, O_X).$$

*Proof.* (i) Let $J$ be an ideal of $A$. Then $C \cdot J$ lies in $C$ and $C$ lies in $O_X$. So $C \cdot J$ is an ideal of $O_X$ that is an $A$-Module.

Let $I$ be an ideal of $O_X$ that is an $A$-Module. Then clearly $I$ lies in $C$. Tensoring the inclusion map $I \to C$ by $C^{-1}$ over $A$ yields an injective map $C^{-1} \otimes_A I \to A$. Its image is an ideal $J$ of $A$.

The assignments, $J \mapsto C \cdot J$ and $I \mapsto \mathrm{Im}(C^{-1} \otimes_A I)$, are mutually inverse because $C \cdot J$ is equal to $C \otimes_A J$ since $C$ is invertible.

(ii) The assertion is obvious.

(iii) Since $C \otimes_A (A/J)$ is equal to $C/C \cdot J$, the assertion is obvious.

(iv) The assertion results immediately from (iii) since $C$ is invertible.

(v) 
$$\begin{aligned} \underline{\mathrm{Hom}}_A(J, A) &= \underline{\mathrm{Hom}}_A(C \cdot J, C) && (C \text{ is invertible} \\ &= \underline{\mathrm{Hom}}_A(C \cdot J, \underline{\mathrm{Hom}}_{U_X}(A, O_X)) && (\text{definition of } C) \\ &= \underline{\mathrm{Hom}}_{O_X}(C \cdot J, O_X) && (\text{a general fact}). \end{aligned}$$

PROPOSITION 4.4. *Let $f\colon X' \to X$ be a finite birational morphism of projective schemes over a locally noetherian base scheme $S$. Set $A = f_* O_{X'}$ and $C = \underline{\mathrm{Hom}}_{O_X}(A, O_X)$. Assume* (a) *$C$ is an invertible $A$-Module and its formation commutes with base change,* (b) *the dual $C \to O_X$ of the comorphism $O_X \to A$ is injective, and* (c) *the sheaf $O_X/C$ is flat over $S$.*

(i) *There is a natural isomorphism from* $\mathrm{Hilb}_{(X'/S)}$ *onto the closed subscheme $Y$ of* $\mathrm{Hilb}_{(X/S)}$ *parameterizing the flat, closed subschemes of $X$ whose ideals are $A$-Modules. It carries a family $W'$ of flat, closed subschemes of $X'/S$ parameterized by an $S$-scheme $T$ onto the family $W$ of flat, closed* *subschemes of $X/S$ parameterized by $T$ whose ideal is given by the formula, $I(W) = C_T \cdot f_* I(W')$.*

(ii) *Let $P'(n)$ be a Hilbert polynomial, set*

$$P(n) = P'(n) + \chi((O_X/C)(n)),$$

*and fix $r \geqslant 0$. Then the isomorphism in* (i) *restricts to an isomorphism from $[\mathrm{Hilb}^{P'(n)}_{(X'/S)}]^r$ onto $Y \cap [\mathrm{Hilb}^{P(n)}_{(X/S)}]^r$.*

*Proof.* (i) Note that $Y$ exists by Proposition 4.2. Now, since $f$ is affine, $f_*$ effects an equivalence of the categories of $O_{X'}$-Modules and $A$-Modules. Consequently, by virtue of Lemma 4.3(i, ii, iv), the map described in (i) is a well-defined isomorphism.

(ii) The assertion follows from Lemma 4.3(iii, v).

LEMMA 4.5. *Let $O$ be the local ring of a node on an irreducible curve over a field $k$. Let $A$ denote the normalization of $O$. Let $I$ be an ideal of $O$. Then either $I$ is principal or else $I$ is an $A$-module.*

*Proof.* (D'Souza [6, Lemma 1.4, p. 47]). Since $A$ is a P.I.D., there is an $f \in I$ such that $fA$ is equal to $IA$. Then the relation $fO \subset I \subset fA$ holds. It follows that either $I$ is equal to $fO$ or else to $fA$ because $\dim_k(A/O)$ is equal to 1.

THEOREM 4.6. *Let $X$ be a projective curve over a field $k$. Let $f\colon X' \to X$ be a map resolving some singular reduced points of $X$. Set $A = f_* O_{X'}$, set $C = \underline{\mathrm{Hom}}_{O_X}(A, O_X)$, and set $\delta = \dim_k \Gamma(O_X/C)$. Fix integers $n$, $r \geqslant 0$.*

(i) *There is a natural closed embedding,*

$$[\mathrm{Hilb}^n_{(X'/k)}]^r \to [\mathrm{Hilb}^{n+\delta}_{(X/k)}]^r.$$

(ii) *Suppose the resolved singularities are all nodes. Then a point $w$ of $[\mathrm{Hilb}^{n+\delta}_{(X/k)}]^r$ lies in the image of $[\mathrm{Hilb}^n_{(X'/k)}]^r$ under the natural embedding if and only if it represents a subscheme $W$ of $W \otimes k(w)$ that is not principal at any of the resolved nodes.*

*Proof.* The dual $C \to O_X$ of the comorphism $O_X \to A$ is bijective off the resolved points. So, its kernel is supported at them. However, there $C$ is torsion free because $O_X$ is. Hence $C \to O_X$ is injective. Moreover, $C$ is an invertible $A$-Module because, as is easy to see, it is isomorphic to $A$ off the resolved points and at them its stalks are isomorphic to ideals in the stalks of $A$, which are P.I.D.'s. The formation of $C$ commutes with base change and the sheaf $O_X/C$ is flat because the base is a field. Therefore Proposition 4.4(ii) implies that there is a natural closed embedding as asserted in (i), and that a point $w$ of the target lies in the image if and only if it represents a subscheme $W$ of $Y \otimes k(w)$ whose ideal is an $A$-Module. If the resolved singularities are nodes, $I(W)$ is an $A$-Module if and only if it is not principal at any of them, by Lemma 4.5.

## 5. Main Results

THEOREM 5.1. *Let $k$ be a field. Assume the conjecture stated in the introduction holds over $k$. Let $X$ be a rational (integral) projective curve with $g$ nodes over $k$. Assume the nodes are in general position. Fix integers $n$, $r \geqslant 0$. Then $[\mathrm{Hilb}^n_{(X/k)}]^r$ has dimension $\leqslant \tau + r$ with $\tau = (r+1)(n-r) - rg$.*

*Proof.* The assertion results immediately from (5.1a) and (5.1b) below.

(5.1a) $[\mathrm{Div}^n_{(X/k)}]^r$ has dimension $\leqslant \tau + r$.

*Proof.* Let $X'$ denote the normalization of $X$, embed it in $\mathbb{P}^n$ as the $n$-ic, and let $M_1, \ldots, M_g$ denote the secant lines to $X'$ determined by the $g$ nodes of $X$. The hypothesis implies $M_1, \ldots, M_g$ are general secants. Hence the conjecture stated in the introduction implies the scheme $R_d$ parameterizing the $[n-r-1]$-planes meeting $M_1, \ldots, M_g$ and containing at least $d$ of them has dimension at most $(r+1)(n-r) - r(g-d) - 2(r+1)d$. Therefore Theorem 3.4(ii) implies by induction and, in its notation, that under the natural map $\gamma$ from $\mathrm{Grass}_{r+1}(E \mid P_n)$ into $\mathrm{Grass}_{n-r-1}(\mathbb{P}^n)$ the inverse image of $R_d$ has dimension at most $\tau - (r+1)d$. Since the image of $\mathrm{Grass}_{r+1}(E \mid P_n)$ is equal to $R_0$ by Theorem 3.4(i), the dimension of $\mathrm{Grass}_{r+1}(E \mid P_n)$ is at most $\tau$. Consequently, Proposition 3.2(iii) implies $[\mathrm{Div}^n_{(X/k)}]^r$ has dimension at most $\tau + r$.

(5.1b) $[\mathrm{Hilb}^n_{(X/k)}]^r - \mathrm{Div}^n_{(X/k)}$ has dimension $\tau + r - 1$.

*Proof.* Construct $X_i'$ by normalizing $X$ at the $i$th node. By Theorem 4.6(i, ii), there is a natural closed embedding of each $[\mathrm{Hilb}^{n-1}_{(X_i'/k)}]^r$ into $[\mathrm{Hilb}^n_{(X/k)}]^r$ and the union of their images is the whole complement of

$\mathrm{Div}^n_{(X/k)}$. We may assume by induction on $n$ or $g$ that $[\mathrm{Hilb}^{n-1}_{(X_i'/k)}]^r$ has dimension,

$$(r+1)(n-1-r) - r(g-1) + r = \tau + r - 1.$$

Thus the assertion holds.

LEMMA 5.2. *Let $k$ be a field. Assume the conjecture stated in the introduction holds over $k$. Then there exists an extension field $K$ of $k$ and a smooth, connected, projective curve $X$ over $K$ such that $[\mathrm{Hilb}^n_{(X/K)}]^r$ has dimension $\leqslant \tau + r$ with $\tau = (r+1)(n-r) - rg$.*

*Proof.* Construct a rational integral projective curve $X_0$ with $g$ nodes in general position over the algebraic closure of $k$. Construct a flat, projective family deforming $X_0$ into a smooth, connected curve $X$ over a suitable extension $K$ of $k$ (the deformation theory involved is discussed in [5, Sect. 1]). By Theorem 5.1, the dimension of $[\mathrm{Hilb}^n_{(X_0/k)}]^r$ is $\leqslant \tau + r$. So, by upper semicontinuity of dimension, Proposition 1.4, the dimension of $[\mathrm{Hilb}^n_{(X/k)}]^r$ is $\leqslant \tau + r$.

THEOREM 5.3. *Let $k$ be an algebraically closed field. Assume the conjecture stated in the introduction holds over $k$. Then most smooth, connected projective curves $X$ of genus $g$ over $k$ are such that each component of $[\mathrm{Hilb}^n_{(X/k)}]^r$ has dimension exactly $\tau + r$ with $\tau = (r+1)(n-r) - rg$; in particular, one exists. In fact, these curves are parameterized by an open, dense subset of the moduli space $\mathcal{M}_g$.*

*Proof.* Consider the family of tricanonically embedded curves; it is smooth and projective over an irreducible parameter space $H_g{}^0$, and its members include every smooth, connected curve over $k$ (see [5, Sect. 3]. By upper semicontinuity of dimension, Proposition 1.4, the members $X$ such that $[\mathrm{Hilb}^n_{(X/K)}]^r$ has dimension at most $\tau + r$, where $K$ is the base field of $X$, are paremeterized by an open subset $U$ of $H_g{}^0$. By Theorem 1.5(i) or (ii), each component of $[\mathrm{Hilb}^n_{(X/K)}]^r$ for each such $X$ has dimension exactly $\tau + r$ because $\mathrm{Hilb}^n_{(X/K)}$ is irreducible with dimension $n$ being isomorphic to the $n$-fold symmetric product [7, Sect. 6; 22; 12]. By Lemma 5.2, the open set $U$ is nonempty, and so dense. Finally, let $h\colon H_g{}^0 \to \mathcal{M}_g$ denote the natural map. It is smooth and surjective (see [5, Sect. 1]. Obviously $U$ is equal to $h^{-1}hU$. Hence $hU$ is open and dense in $\mathcal{M}_g$.

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## APPENDIX

*Dan Laksov*

The purpose of the first part of this Appendix is to prove that the second part of the conjecture of the introduction follows from the first part. In the second part of the Appendix we prove the conjecture when $r = 1$. Moreover, when $r = 1$, $g = 2(n - 1)$ and the characteristic is zero, we prove that the subset $T$ of the conjecture consists of exactly

$$N = \frac{g!}{(g - n + 1)!\,(g - n + 2)!} = \frac{(2(2 - 1))!}{(n - 1)!\,n!}$$

simple points.

Then, by Theorem 5.3, most smooth connected projective curves $X$ of genus $g$ are such that each component of $[\mathrm{Hilb}^n_{X|k}]^1$ has dimension exactly $2(n - 1) - g + 1$. In particular, since such components exists, we obtain a new proof of the existence of special divisors in the case $r = 1$. Moreover, when $g = 2(n - 1)$ and the characteristic is zero, it follows immediately, from the techniques used in the proofs of Theorems 5.1 and 5.3, that there are, on most smooth projective connected curves of genus $g$, at least $N$ distinct divisor classes $g_n{}^1$ of degree $n$ and projective dimension at least one. However, it is known ([6], [3, Corollary to Theorem 3, p. 15], or [7, Proposition 4, p. 168], which is valid in all characteristics) that there are exactly $N$ such divisor classes when counted with multiplicity. Consequently, on a general curve there are exactly $N$ distinct divisor classes $g_n{}^1$ each with multiplicity one. This result was first asserted by Castelnuovo [1, Sect. 3, p. 68].

Note that even in characteristic zero it is not true that the divisor classes $g_n{}^1$ are all distinct. Indeed, on a general curve of genus four there are, by the above result, two distinct divisor classes $g_3^1$. However, let $X$ be the intersection in $\mathbb{P}^3$ of a quadric cone $F$ whose base is a nonsingular conic and a general cubic hypersurface. Then by Bertini's theorem $X$ is a nonsingular curve, and it is of degree six by Bezout's theorem. Hence, by the genus formula for a complete intersection [14, Chap. IV, Sect. 2–7, formula (18), p. 73] $X$ has genus four. The hyperplanes in $\mathbb{P}^3$ induce on $X$ a linear system of projective dimension three and of degree six. Hence the embedding of $X$ in $\mathbb{P}^3$ is the canonical embedding. Let $D$ be a divisor on $C$ belonging to a $g_3^1$. Then there are at least two independent differentials $\omega_1$ and $\omega_2$ on $C$ vanishing on $D$. Corresponding to $\omega_1$ and $\omega_2$ there are two independent planes $L_1$ and $L_2$ in $\mathbb{P}^3$ containing the divisor $D$. The line $l$ of intersection of the two planes contains the divisor $D$ and hence intersects the quadric cone $F$ in at least three points. Consequently, $l$ must be one of the generators of the cone $F$. Thus the divisors of $C$ belonging to any $g_3^1$ correspond to the generators of $F$. Consequently, they are all linearly equivalent and there is only one $g_3^1$.

*Part* 1

Let $C$ be a smooth rational curve of degree $n$ in $\mathbb{P}^n$, not contained in any hyperplane.

LEMMA 1. *Let $P_0, P_1, \ldots, P_t$ be $t$ points of $C$. If distinct, they span a linear space of dimension* $\min(n, t)$.

*Proof.* Suppose that the points span a linear space $L$ of dimension $s < \min(n, t)$. Since $C$ is not contained in any hyperplane of $\mathbb{P}^n$, we can inductively choose a point $Q_{i+1}$ of $C$ outside the linear space $L_i$ spanned by $L$ and $Q_1, Q_2, \ldots, Q_i$ for $i = 1, \ldots, n - s - 1$. Then the hyperplane $L_{n-s-1}$ intersects $C$ in at least $(n + t - s) > n$ distinct points, namely, $P_0, \ldots, P_{t'}\, Q_1, \ldots, Q_{n-s-1}$. This is impossible by Bezout's theorem.

LEMMA 2. *Let $X$ be an irreducible curve in $\mathbb{P}^n$ of degree at most $n$ and not contained in any hyperplane. Then $X$ is a smooth, rational curve of degree $n$ in $\mathbb{P}^n$.*

*Proof.* The case $r = 1$ is trivial. We proceed by induction on $n$. Choose a point $P$ on the curve $X$, and consider the projection of $\mathbb{P}^n$ to $\mathbb{P}^{n-1}$ with center $P$. Denote by $X'$ the image of the curve $X$ and by $n'$ its degree. Let $H'$ be a hyperplane in $\mathbb{P}^{n-1}$ which intersects $X'$ in $n'$ simple points $Q_1, \ldots, Q_n$, that are images of points $P_1, \ldots, P_{n'}$ on $X$ different from $P$. Then the hyperplane $H$ in $\mathbb{P}^n$ determined by $P$ and $H'$

intersects $X$ in the points $P, P_1, \ldots, P_{n'}$. Since $X$ is of degree at most $n$, we must have $(n' + 1) \leqq n$. By induction, therefore, $X'$ is a smooth, rational curve of degree $n' = (n - 1)$ in $\mathbb{P}^{n-1}$.

By hypothesis, the degree of $X$ is at most $n$. On the other hand, the hyperplane $H$ intersects $X$ in exactly the $n = n' + 1$ points $P, P_1, \ldots, P_n$. Hence $X$ has degree $n$ in $\mathbb{P}^n$ and the points appear with multiplicity one. Therefore, $P$ is a nonsingular point of $X$. Since $P$ was chosen arbitrarily, $X$ is smooth.

Finally, the fiber over the point $Q_1$ of the morphism $\pi$ from $X$ to $X'$ induced by the projection clearly consists of the point $P_1$ alone. Moreover, it is easily verified that the multiplicity of the fiber at $P_1$ is bounded by the intersection multiplicity of $X$ and $H$ at $P_1$ and thus equal to one. Consequently, the morphism $\pi$ is birational. Hence, $X$ is a rational curve since $X'$ is.

LEMMA 3. *Denote by $G(n - r - 1, n)$ the grassmannian of $[n - r - 1]$-planes in $\mathbb{P}^n$. Let $L$ be a $t$-plane in $P^n$. Then there is a natural isomorphism between the closed subscheme of $G(n - r - 1, n)$ representing the $[n - r - 1]$-planes containing $L$ and the grassmannian $G(n - t - r - 1, n - t)$.*

*Proof.* The definition of the natural morphism as well as the verification of the assertion of Lemma 3 are obvious.

THEOREM 4. *Assume that the $[n - r - 1]$-planes in $\mathbb{P}^n$ intersecting $g$ general secants of the curve $C$ are parameterized by a closed subset of $G(n - r - 1, n)$ of dimension at most $\tau = (r + 1)(n - r) - rg$. Then the $[n - r - 1]$-planes in $\mathbb{P}^n$ which in addition contain $d$ or more of the secants are parameterized by a closed subset of dimension at most $\tau - (r + 2)d$.*

*Proof.* Notice that for $2d > (n - r)$, the conclusion holds even without the hypothesis. Indeed, this inequality implies $\tau - d(r + 2) < r(d - g)$, and consequently $\tau - d(r + 2) < 0$. On the other hand, choose $d$ secants to $C$ which pass through at least $2d$ different points of $C$. Then by Lemma 1 a linear space that contains the $d$ secants must have dimension at least $(2d - 1)$. Since $(2d - 1) > (n - r - 1)$, there is consequently no $[n - r - 1]$-plane containing the $d$ secants.

We may thus assume $2d \leqq (n - r)$. Choose $2d$ different points on the curve and divide them into $d$ pairs. Denote by $L$ the linear space spanned by the $2d$ points and by $s_1, \ldots, s_d$ the $d$ secants determined by the $d$ pairs of points. By Lemma 1, $L$ is of dimension $(2d - 1)$. Consider the projection from $\mathbb{P}^n$ to $\mathbb{P}^{n-2d}$ with center $L$ and denote by $C'$ the image of $C$. By Lemma 3 there is a natural isomorphism between the space parameterizing the $[n - r - 1]$-planes in $\mathbb{P}^n$ containing the $d$ secants $s_1, \ldots, s_d$ and the grassmannian paramaterizing $[n - 2d - r - 1]$-planes in $\mathbb{P}^{n-2d}$.

A $[n - r - 1]$-plane in $\mathbb{P}^n$ intersecting a secant $s$ of $C$ which does not pass through $L$ maps to a $[n - 2d - r - 1]$-plane in $\mathbb{P}^{n-2d}$ which intersects the image of $s$ under the above projection. Conversely, let $s'$ be a secant to $C'$ in general position. Then clearly there are a finite number of secants $t_1, \ldots, t_k$ to $C$ which map to $s'$ under the projection, and none of these secants intersect $L$. An $[n - 2d - r - 1]$-plane intersecting $s'$ will correspond to an $[n - r - 1]$-plane intersecting all of the secants $t_1, \ldots, t_k$. We conclude that the $[n - r - 1]$-planes containing the $d$ secants $s_1, \ldots, s_d$ and intersecting $(g - d)$ general secants of $C$ depend on just as many parameters as the $[n - 2d - r - 1]$-planes of $\mathbb{P}^{n-2d}$ intersecting $(g - d)$ general secants of the curve $C'$.

Assume that $C'$ is a rational, nonsingular curve not contained in any hyperplane of $\mathbb{P}^{n-2d}$. Then by hypothesis the $[n - 2d - r - 1]$-planes in $\mathbb{P}^{n-2d}$ intersecting $(g - d)$ general secants of $C'$ depend on at most

$$(n - 2d - r)(r + 1) - r(g - d) = \tau - d(r + 2)$$

parameters. Consequently, the conclusion of Theorem 4 holds for the curve $C$.

The curve $C'$, being a projection of the curve $C$, clearly is not contained in any hyperplane in $\mathbb{P}^{n-2d}$. To prove that $C'$ is a rational, nonsingular curve it is, by Lemma 2, sufficient to prove that its degree $d'$ is at most $(n - 2d)$. Let $H'$ be a hyperplane intersecting $C'$ in $d'$ distinct points, all of which are images of points on $C$ and not on $L$. Then the hyperplane $H$ in $\mathbb{P}^n$ determined by $H'$ and $L$ intersects $C$ in at least $d'$ points outside of $L$, and it contains the $2d$ points on $C$ spanning $L$. Since $C$ is of degree $n$, we have $(d' + 2d) \leqq n$. Thus, the theorem is proven.

*Part* 2

We shall keep the notation of Part 1. Most of this section is devoted to a proof of the following result.

THEOREM 5. *The $[n - 2]$-planes in $\mathbb{P}^n$ that intersect $g$ general secants to the curve $C$ are parameterized by a closed subset $T$ of $G(n - 2, n)$ of dimension exactly* $2(n - 1) - g$ *(empty if* $2(n - 1) - g < 0$).

*Assume that the ground field is of characteristic zero. Then the components of $T$ all occur with multiplicity one. Moreover, when $g = 2(n-1)$, then $T$ consists of exactly*

$$\frac{g!}{(g-n+1)!\,(g-n+2)!}$$

*points.*

Let $l$ be a line in $\mathbb{P}^n$. Denote by $\sigma(l)$ the Schubert subscheme of $G(n-2, n)$ representing $[n-2]$-planes intersecting $l$. It is well known (see, e.g., [4, Remark (2.4(iv), p. 286]) that $\sigma(l)$ is an irreducible hyperplane section in $G(n-2, n)$.

LEMMA 6. *Fix a point $(L)$ of $G(n-2, n)$. Then a general member of the family of Schubert schemes, $\{\sigma(l) \mid l$ a secant to $C\}$ does not contain $(L)$.*

*Proof.* It is clearly sufficient to prove that not all secants of the curve $C$ pass through any given $[n-2]$-plane $L$. However, if so, the curve would be contained in the hyperplane spanned by $L$ and any point of $C$.

The first part of Theorem 5 is an immediate consequence of Lemma 6. Indeed, by the lemma, we may choose inductively lines $l_1, l_2, \ldots$ in $\mathbb{P}^n$ such that $\sigma(l_i)$ avoids a finite set of points, one on each component of the intersection $\sigma(l_1) \cap \cdots \cap \sigma(l_{2-1})$. Since $\sigma(l_i)$ is a hyperplane section in $G(n-2, n)$, the scheme $\sigma = \sigma(l_1) \cap \cdots \cap \sigma(l_g)$ has pure codimension $g$ in $G(n-2, n)$ (empty if $g > 2(n-1)$).

In the following, we shall assume that the characteristic is zero. Denote by $C(2)$ the second symmetric power of the curve $C$. Then the embedding of $C$ in $\mathbb{P}^n$ determines an embedding of $C(2)$ in $G(1, n)$ [13, Proposition 1, p. 625]. Intuitively, the embedding sends an unordered pair of points $(P, Q)$ on $C$ to the point in $G(1, n)$ representing the secant line through $P$ and $Q$ (when $P = Q$, the secant is the tangent through $P$).

Denote by $W$ the scheme-theoretic intersection of the subscheme $C(2) \times G(n-2, n)$ of $G(1, n) \times G(n-2, n)$ with the universal subscheme of $G(1, n) \times G(n-2, n)$ representing pairs $(l, L)$ where $l$ is a line in $\mathbb{P}^n$ intersecting the $[n-2]$-plane $L$. Denote by $p$ and $q$ the morphisms from $W$ to $C(2)$ and $G(n-2, n)$ induced by the projection of $C(2) \times G(n-2, n)$ onto the first and second factors.

$$\begin{array}{ccc} & W & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} \\ C(12) & & G(n-2, n) \end{array}$$

The points of $W$ represent the pairs $(l, L)$ where $l$ is a secant line to $C$ and $L$ is an $[n-2]$-plane intersecting $L$. Denote by $W_0$ the closed subscheme of $W$ representing the pairs $(l, L)$ where the line $l$ is contained in the $[n-2]$-plane $L$.

LEMMA 7. *Let $l$ and $L$ be a line and an $[n-2]$-plane in $\mathbb{P}^n$. Then:*

(i) *The morphism $q$ induces an isomorphism of the fiber of $p$ at the point of $C(2)$ determining $l$ onto the Schubert subscheme $\sigma(l)$ of $G(n-2, n)$.*

(ii) *The morphism $p$ induces an isomorphism of the fiber of $q$ at the point of $G(n-2, n)$ representing $L$ onto the intersection of the subscheme $C(2)$ of $G(1, n)$ with the Schubert subscheme $\sigma(L)$ of $G(1, n)$ (representing lines in $\mathbb{P}^n$ intersecting $L$).*

*Proof.* The statements of Lemma 7 are immediate consequences of the corresponding well-known properties of the universal subscheme of $G(1, n) \times G(n-2, n)$ representing the pairs $(l, L)$ where $l$ is a line intersecting the $[n-2]$-plane $L$.

LEMMA 8. *The morphism $q$ is flat and the restriction $q \mid (W - W_0)$ of $q$ to the open subsheme $(W - W_0)$ of $W$ is generically smooth.*

*Proof.* The Schubert schemes $\sigma(l)$ are hyperplane sections of $G(n-2, n)$; hence, $W$ is a divisor in $C(2) \times G(n-2, n)$. Let $L$ be an $[n-2]$-plane. Then the points of the fiber $q^{-1}(L)$ of $q$ at the point representing $L$ correspond to secants to $C$ intersecting $L$. It follows from Lemma 6 that $q^{-1}(L)$ is properly contained in $C(2)$. Since $C(2)$ is integral, $q^{-1}(L)$ is therefore a divisor. Since the projection of $C(2) \times G(n-2, n)$ onto $G(n-2, n)$ is flat, we conclude that its restriction $q$ is flat [10, Lecture 10, Proposition–Definition, p. 72].

The projective linear group $\mathrm{PGL}(n+1)$ operates in a natural way on $G(1, n)$. The action is transitive and an element $\alpha$ in $\mathrm{PGL}(n+1)$ sends a Schubert scheme $\sigma(L)$ isomorphically onto the Schubert scheme $\sigma(\alpha L)$. We conclude [5, Corollary 4(ii), p. 291] that for a general $[n-2]$-plane $L$ in $\mathbb{P}^n$ the intersection of $\sigma(L)$ with $C(2)$ will be smooth outside of the singular locus of $\sigma(L)$. The singular locus of $\sigma(L)$ is the closed subscheme $\sigma_0(L)$ representing lines contained in $L$ [9, Corollary (6.3), p. 428]. Moreover, by Lemma 7(ii), the intersection $C(2) \cap (\sigma(L) - \sigma_0(L))$ is isomorphic to the fiber of $q \mid (W - W_0)$ at the point of $G(n-2, n)$ representing $L$. Consequently, $q \mid (W - W_0)$ is generically smooth.

PROPOSITION 9. *Let $Z$ be an irreducible subscheme of $G(n-2, n)$ and $D$ be the subscheme of $G(n-2, n)$ over which the restriction $q \mid (W - W_0)$ is not smooth. Then a general member of the family $\{\sigma(l) \mid l$ a secant to the curve $C\}$ intersects $Z$ properly, and the intersection is transversal outside of $D$ and of the singular loci of $\sigma(l)$ and $Z$.*

*Proof.* Consider the diagram

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & & Z \arrow[dl, "i"'] \\ C(2) & & G(n-2, n) & \end{tikzcd}$$

where $i$ is the inclusion. Since $q$ is a flat morphism (Lemma 8) it follows from a general transversality result [5, Lemma 1(i), p. 288] that for each point $(l)$ in an open dense subset of $C(2)$ the fibered product $p^{-1}(l) \times_{G(n-2,n)} Z$ is empty or equidimensional of dimension, $\dim p^{-1}(l) + \dim Z - \dim G(n-2, n)$. By Lemma 7(i), the fibered product is isomorphic to the intersection $\sigma(l) \cap Z$ in $G(n-2, n)$. The first part of the proposition follows immediately.

Denote by $W'$ the complement of the union $W_0 \cup q^{-1}(D)$ in $W$ and by $Z'$ the complement of the singular locus of $Z$. Then we have a diagram

$$\begin{tikzcd} & W' \arrow[dl, "p'"'] \arrow[dr, "q'"] & & Z' \arrow[dl, "i'"'] \\ C(2) & & G(n-2, n) & \end{tikzcd}$$

where $p'$, $q'$, and $i'$ are the morphisms induced by $p$, $q$, and $i$. From the transversality result referred to above [5, Lemma 1(ii), p. 288], it follows that for each point $(l)$ in an open subset of $C(2)$ the fibered product $(p')^{-1}(l) \times_{G(n-2,n)} Z'$ is smooth. By Lemma 7(i) the fibered product is isomorphic to the intersection $(\sigma(l) - \sigma_0(l)) \cap Z' \cap (G(n-2, n) - D)$, where $\sigma_0(l)$ is the closed subscheme of $\sigma(l)$ representing $[n-2]$-planes containing $l$. The second part of the proposition follows, since $\sigma_0(L)$ is the singular locus of $\sigma(L)$ [9, Corollary (6.3), p. 428].

We now establish the second part of Theorem 5. By Lemma 8, the set $D$ of Proposition 9 is a proper subset of $G(n-2, n)$. Hence Proposition 9 allows us to choose inductively secants $l_1, l_2, \ldots$ to the curve such that (1) $\sigma(l_i)$ intersects the two schemes, $\sigma(l_1) \cap \cdots \cap \sigma(l_{i-1})$ and $D \cap \sigma(l_1) \cap \cdots \cap \sigma(l_{i-1})$, properly and (2) the intersection of $\sigma(l_i)$ with each of these schemes is smooth outside of $D$ and their singular loci. Then all components of the intersection $\sigma(l_1) \cap \cdots \cap \sigma(l_g)$ are of codimension $g$ in $G(n-2, n)$ and occur with multiplicity one.

Assume that $g = 2(n-1)$; then the intersection $\sigma(l_1) \cap \cdots \cap \sigma(l_g)$ consists of a finite number $N$ of points of $G(n-2, n)$, each with multiplicity one. Consider $G(n-2, n)$ as a closed subscheme of the $M = \binom{n+1}{n-1}$-dimensional projective space $\mathbb{P}^M$. Then the Schubert schemes $\sigma(l_i)$ are intersections of $G(n-2, n)$ with certain hyperplanes in $\mathbb{P}^M$ [8, Corollary 5, p. 1068]. Consequently, the number $N$ is equal to the degree of the scheme $G(n-2, n)$. This number was determined by Schubert [12, formula (26), p. 117] using Pieri's formula. He found that

$$N = \frac{g!}{(g-n+1)!\,(g-n+2)!} = \frac{(2(n-1))!}{(n-1)!\,n!}$$

(see also [2, Vol. II, Chap. VIX, Sect. 7, formula (9), p. 366]. This finishes the proof of Theorem 5.

*Remark.* In characteristic zero, the $[n-2]$-planes in $\mathbb{P}^n$ that intersect $g$ general tangents to the curve $C$ are parameterized by a closed subscheme $T$ of $G(n-2, n)$ of dimension exactly $2(n-1) - g$ (empty for $2(n-1) - g < 0$) and all the components of $T$ occur with multiplicity one. Indeed it is easy to prove results similar to Lemma 8 and Proposition 9 above, involving instead of the inclusion of $C(2)$ in $G(n-2, n)$, the natural inclusion of $C$ in $G(n-2, n)$ which maps a point $P$ of $C$ to the point representing the tangent line to $C$ at $P$. The crucial tool in the proof is the result that an $[n-2]$-plane $L$ in $\mathbb{P}^n$ does not intersect an infinite number of tangents to the curve $C$. This can be seen by projecting $C$ from the center $L$ onto the line $\mathbb{P}^1$. Then each tangent to the curve $C$ which intersects $L$ corresponds to a point of ramification of the projection. However, in characteristic zero, it is well-known that there is at most a finite number of points of ramification.

The assumption on the characteristic is necessary. Indeed, let $C$ be the twisted cubic in $\mathbb{P}^3$ and assume that the characteristic is two. Project the curve $C$ from a point $P$ on $C$ onto a curve $C'$ in $\mathbb{P}^2$. Then, clearly, $C'$ is a curve of degree two in $\mathbb{P}^2$ and consequently is a strange curve [11, Chap. II, Sect. 1, Definition, p. 46]. Let $P'$ be a point of $\mathbb{P}^2$ through which all the tangents to $C$ pass. Then all tangents to $C$ will intersect the line in $\mathbb{P}^3$ through $P$ determined by $P'$. Hence, moving the point $P'$, we obtain a pencil of lines such that each member of the pencil intersects all the tangents to $C$. We conclude that the lines that intersect any $g$ tangents to the curve $C$ depend on at least one parameter, whereas, in characteristic zero, the number of parameters is at most zero when $g \leqslant 4$.

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